The Substitution Rule (Change of Variables)

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A commonly used technique for integration is Change of Variable, also called Integration by Substitution.

Recall the Chain Rule for differentiation: If $y = F(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dF}{du}(g(x))\frac{dg}{dx}(x)$$

The above implies that

$$dy = dF(g(x)) = F'(g(x))g'(x)\,dx = F'(g(x))\,dg(x) = F'(u)\,du$$

By letting $f = F'$, we thus obtain the following theorem:

Theorem 1. *(Change of Variables) If $f(u)$ is continuous on an interval I and $u = g(x)$ is a differentiable function whose range is in I, then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Remark 2. There are two possible situations to apply the method of change of variable. The first case is when we can write the given integral in the form of $\int f(g(x))g'(x)\,dx$ by a good choice of $g(x)$, and then apply the formula. See Example 4,5,6

The second case is to use the theorem in the reversed way: let x be the intermediate variable by letting $x = g(t)$, so $\int f(x)\,dx = \int f(g(t))g'(t)\,dt$. See Example 7

Remark 3. We can also write the change of variable in differential notation:

$$\int f(g(x))\,dg(x) = \int f(u)\,du$$

where $u = g(x)$

Example 4. *Compute* $\int x^2 + 2x + 1\, dx$

$$\begin{aligned}\int x^2 + 2x + 1\, dx &= \int (x+1)^2\, dx \\ &= \int (x+1)^2\, d(x+1) \\ &= \frac{1}{3}(x+1)^3 + C\end{aligned}$$

Example 5. *Compute* $\int x\sqrt{1+x^2}\, dx$

$$\begin{aligned}\int x\sqrt{1+x^2}\, dx &= \int \frac{1}{2}\sqrt{1+x^2}\, d(1+x^2) \\ &= \frac{1}{2}\int \sqrt{1+x^2}\, d(1+x^2) \\ &= \frac{1}{2} \times \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C \\ &= \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C\end{aligned}$$

Example 6. *Compute* $\int x^2 \cos(x^3 - 1)\, dx$

$$\begin{aligned}\int x^2 \cos(x^3-1)\, dx &= \frac{1}{3}\int \cos(x^3-1)\, d(x^3-1) \\ &= \frac{1}{3}\sin(x^3-1) + C\end{aligned}$$

Example 7. *Compute* $\int \ln x\, dx$

Let $x = e^t$, *so* $t = \ln x$.

$$\begin{aligned}\int \ln x\, dx = \int \ln e^t\, de^t &= \int t de^t \\ &= te^t - \int e^t\, dt \\ &= te^t - e^t + C \\ &= x\ln x - x + C\end{aligned}$$

Example 8. *Calculate* $\int \tan x\, dx$

$$\begin{aligned}\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx \\ &= -\int \frac{1}{\cos x}\, d\cos x \\ &= -\ln|\cos x| + C\end{aligned}$$

There is also a corresponding formula for definite integrals:

Theorem 9. *If $f(u)$ is continuous and $u = g(x)$ is continuously differentiable, then*

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof. Let F be an antiderivative of f, then

$$\int_a^b f(g(x))g'(x)\, dx = F(g(x))\Big|_a^b = F(u)\Big|_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\, du$$

□

Example 10. *Evaluate* $\int_1^e \frac{1+\ln x}{x}\, dx$
There are different alternatives to apply the rules:

Method I:

$$\begin{aligned}\int_1^e \frac{1+\ln x}{x}\, dx &= \int_1^e (1+\ln x)\, d\ln x \\ &= \int_{\ln 1}^{\ln e} (1+u)\, du \\ &= \int_0^1 (1+u)\, du \\ &= u + \frac{1}{2}u^2\Big|_0^1 \\ &= \frac{3}{2}\end{aligned}$$

Method II:

$$\begin{aligned}\int_1^e \frac{1+\ln x}{x}\,dx &= \int_1^e (1+\ln x)\,d(1+\ln x)\\ &= \int_{1+\ln 1}^{1+\ln e} u\,du\\ &= \int_1^2 u\,du\\ &= \frac{1}{2}u^2\Big|_1^2\\ &= \frac{3}{2}\end{aligned}$$

Method III: Let $x = e^t$

$$\begin{aligned}\int_1^e \frac{1+\ln x}{x}\,dx &= \int_0^1 \frac{1+\ln e^t}{e^t}\,de^t\\ &= \int_0^1 \frac{1+t}{e^t}(e^t)'\,dt\\ &= \int_0^1 \frac{1+t}{e^t}e^t\,dt\\ &= \int_0^1 1+t\,dt\\ &= \frac{3}{2}\end{aligned}$$

Example 11. *Evaluate $\int_0^1 \sqrt{1+x}\,dx$*

$$\begin{aligned}\int_0^1 \sqrt{1+x}\,dx &= \int_0^1 \sqrt{1+x}\,d(1+x)\\ &= \int_1^2 \sqrt{u}\,du\\ &= \frac{4}{3}\sqrt{2}-\frac{2}{3}\end{aligned}$$

Example 12. *Compute $\int_0^1 x^3\sqrt{1+x^2}\,dx$*

Method I:

$$\begin{aligned}
\int_0^1 x^3\sqrt{1+x^2}\,dx &= \frac{1}{2}\int_0^1 x^2\sqrt{1+x^2}\,d(1+x^2)\\
&= \frac{1}{2}\int_0^1 ((1+x^2)-1)\sqrt{1+x^2}\,d(1+x^2)\\
&= \frac{1}{2}\int_{1+0^2}^{1+1^2} (u-1)u^{\frac{1}{2}}\,du\\
&= \frac{1}{2}\int_1^2 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du\\
&= \frac{1}{2}(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}})\Big|_1^2 = \frac{2}{15} + \frac{2}{15}\sqrt{2}
\end{aligned}$$

There is another way of writing the above computation, though the reason behind are the same:

Let $u = 1 + x^2$*, then* $du = d(1+x^2) = (1+x^2)'du = 2xdx$

$$\begin{aligned}
\int_0^1 x^3\sqrt{1+x^2}\,dx &= \int_{1+0^2}^{1+1^2} x^3\sqrt{u}\frac{1}{2x}\,du\\
&= \int_1^2 \frac{x^2}{2}\sqrt{u}\,du\\
&= \int_1^2 \frac{u-1}{2}\sqrt{u}\,du\\
&= \frac{1}{2}\int_1^2 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du\\
&= \frac{1}{2}(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}})\Big|_1^2\\
&= \frac{2}{15} + \frac{2}{15}\sqrt{2}
\end{aligned}$$

Method II: Let $x = \sqrt{t^2-1} = (t^2-1)^{\frac{1}{2}}$

$$\begin{aligned}
\int_0^1 x^3\sqrt{1+x^2}\,dx &= \int_1^{\sqrt{2}} ((t^2-1)^{\frac{1}{2}})^3(1+((t^2-1)^{\frac{1}{2}})^2)^{\frac{1}{2}}\,d(t^2-1)^{\frac{1}{2}} \\
&= \int_1^{\sqrt{2}} (t^2-1)^{\frac{3}{2}}(1+t^2-1)((t^2-1)^{\frac{1}{2}})'\,dt \\
&= \int_1^{\sqrt{2}} (t^2-1)^{\frac{3}{2}}t(\frac{1}{2}(t^2-1)^{-\frac{1}{2}}2t)\,dt \\
&= \int_1^{\sqrt{2}} t^2(t^2-1)\,dt \\
&= \int_1^{\sqrt{2}} t^4-t^2\,dt \\
&= \frac{t^5}{5}-\frac{t^3}{3}\Big|_1^{\sqrt{2}} \\
&= \frac{2}{15}+\frac{2}{15}\sqrt{2}
\end{aligned}$$

Corollary 13. *f is a continuous function on the interval $[-a,a]$.*

(i). If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$;

(ii). If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

Proof. $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$

(i). If f is even, $\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} f(-x)\,dx = \int_a^0 f(u)d(-u) = \int_0^a f(u)\,du$

So $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

(ii). If f is odd, $\int_{-a}^{0} f(x)\,dx = -\int_{-a}^{0} f(-x)\,dx = -\int_a^0 f(u)d(-u) = -\int_0^a f(u)\,du$

So $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0$ □